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Division of Fractions: Preservice Teachers' Understanding and Use of Problem Solving Strategies

Leah Nillas
Illinois State University, Normal, Illinois

Abstract: *A teacher's profound mathematical understanding is critical in learning to teach mathematics for understanding (Fennema & Franke, 1992; Ma, 1999). This research study investigated preservice teachers' solution processes to determine their conceptual and procedural understanding of division of fractions. Qualitative methods, involving interviews and analysis of written responses on problem solving tasks were employed. Ten preservice teachers responded to five problems and four of them were interviewed to clarify their problem solving strategies. Results showed that preservice teachers used a variety of strategies but lacked a strong conceptual understanding of the meaning of division of fractions. Findings supported the need to provide a strong content preparation for preservice teachers with focus on teaching and learning for understanding.*

Carpenter and Lehrer (1999) argued that "unless students learn with understanding, whatever knowledge they acquire is likely to be of little use to them outside the school" (p. 20). For preservice teachers, Ball (1991) specifically addressed the significance of the subject matter knowledge with emphasis on how to develop understanding of the content. She focused on teachers' deep understanding of the mathematics content and the underlying principles and meanings behind mathematical ideas. Consistent with Ball, Leinhardt, Putnam, Stein, and Baxter (1991) investigated how the nature and organization of teachers' subject matter knowledge influences their teaching.

Similarly, several studies supported the notion that teachers' knowledge has a huge impact on students' learning (Ball, 1991; Even, 1993; Leinhardt et. al., 1991). Teachers' subject matter knowledge shapes the ways in which they teach mathematics. Also, teachers' subject matter knowledge interacts with their assumptions, explicit beliefs about teaching and learning, about students and about contexts (Ball). In-depth understanding of the subject matter will enable teachers to show the connection of mathematical ideas to students (Ball, 1990b). Since knowledge develops based on the teacher's pedagogical knowledge and through classroom interactions with the subject matter and the students then the teacher's knowledge plays a critical role in student's learning (Fennema & Franke, 1992). As a consequence, transformation of knowledge should be viewed as an important goal in teaching. In this context, teaching should aim for both the conceptual and procedural understanding.

With respect to the teaching and learning of division of fractions, while most studies have focused on students' understanding of division of fractions (e.g., Alkhateeb & Nicholls, 2001; Bezuk & Armstrong, 1993; Huinker, 1998; Nowlin, 1996; Ottino & Snook, 1991; Warrington, 1997; Lewis, 1996; Mack, 1990, 1995; Sharp, 1998; Siebert, 2002; Streefland, 1993) others have focused on preservice teachers' understanding of division of fractions (e.g., Ball, 1990a; Borko et. al., 1992; Khoury & Zazkis, 1994; Lubinski, Fox, & Thomason, 1998; Ma, 1999; Tirosh, 2000). However, none of these studies examined the strategies used by preservice teachers given different division-of-fraction situations. This paper thus focuses on these strategies-in-context and aims to contribute to the knowledge base on preservice teachers' teaching and learning with understanding. In particular, the purpose of this study is to investigate preservice teachers' strategies and understanding of division of fractions in solving problems.

To address this research goal, the following questions were investigated and addressed: (1) What strategies do preservice teachers use in solving division of fraction problems? (2) Do these strategies differ depending on division of fraction situations? Throughout this paper, I used the term *understanding* to refer to both forms of knowledge, conceptual and procedural, which is characterized by comprehension of mathematical concepts and fluency in the use of procedures.

Preservice Teachers' Understanding

Ball (1990a) found that preservice teachers had significant difficulty with the meaning of division of fractions. In her study, Ball examined the mathematical and pedagogical knowledge and reasoning of nineteen preservice teachers about division. Specifically, she focused on preservice teachers' understanding of division of fractions, division by zero, and division with algebraic equations. She found that most preservice teachers perceived division by a fraction problem in terms of fractions and not as a division problem. Also, they had difficulty relating the fractional expression, $1\frac{3}{4} \div \frac{1}{2}$ to a real life situation. Moreover, most of them used a procedural solution and justified using the "can't divide by zero rule" when given problems involving division with an algebraic equation and division by zero, respectively.

Two years after Ball's study, Borko and others (1992) presented an analysis of a classroom lesson where a preservice teacher failed to provide a conceptual justification for the division of fraction algorithm for the expression $\frac{3}{4} \div \frac{1}{2}$. In contrast to Ball's study, Borko and her colleagues focused on a preservice teacher, Ms. Daniels, on her beliefs about good mathematics teaching and about learning to teach, and about her knowledge of division of fractions. Though Ms.Daniels' beliefs about good teaching were consistent with the current views about effective teaching, it was difficult for her pedagogically to express them because she did not have a strong conceptual knowledge base for division of fractions.

Similar to the work of Borko and her colleagues, Lubinski and her colleagues (1998) focused on a single preservice teacher, Rebecca. They presented Rebecca's case to exemplify that preservice teachers can be provided with a way to better understand the mathematics they will teach. In particular, Rebecca was provided with an opportunity to develop in-depth understanding by reflecting on her own reasoning and sense-making processes as she developed the meaning behind the expression $\frac{2}{3} \div \frac{5}{7}$. Lubinski and her colleagues showed and argued that preservice teachers need to find meaning within their own reasoning processes. They also recognized that the role of social context is important in preservice teachers' sense-making.

With a different focus, Khoury and Zazkis (1994) examined 124 preservice teachers' reasoning strategies and arguments in solving fraction problems in different symbolic representations. They found that the majority of preservice teachers believed that fractions change their numerical value under different symbolic representations. Although their focus was not specific to division of fractions, the analysis of their findings indicated and supported that "preservice teachers' knowledge of place value and rational numbers is more syntactical than conceptual" (p. 203). As a corollary to this, Ma (1999) argued that a teacher should first have a profound understanding of the concept in order to have a pedagogically powerful representation of fractions. With data from Chinese and American preservice teachers, Ma found that most preservice teachers had weak conceptual understanding of division of fractions and had difficulty relating division context to real life situations.

Tirosh (2000), on the other hand, discussed how most preservice teachers can divide fractions but could not explain why the procedure works. Different from other studies, Tirosh presented an argument of how to promote development of preservice teachers' subject matter knowledge of division of fractions and their awareness to sources of students' common misconceptions as well as how to address them. She identified students' sources of misconceptions based on the algorithm, intuition, and formal knowledge as categorized by previous literature. Her findings strongly supported the need to familiarize preservice teachers with students' common erroneous cognitive processes to enable them to respond to their misconceptions.

From the studies discussed above, it is clear that many preservice teachers have poor conceptual understanding of division of fractions and thus the need to address this issue is paramount. Different from the other studies, this current study provides preservice teachers with five different problem situations involving division of fractions. Moreover, this study focuses on preservice teachers' strategies and conceptual understanding which are assessed through problem-solving and problem-posing tasks, respectively.

Methodology

Participants

Ten elementary preservice teachers, chosen based on their willingness to participate, took part in this study. All but one had taken a problem-solving class which covered fractions as a focused content topic and all were at different stages or levels of their academic program. Four of the ten were purposely selected for in-depth interview to further probe their strategies and understanding of division of fractions.

Procedure

The ten preservice teachers were asked to answer five problem-solving questions designed to assess their conceptual and procedural understanding on division of fractions. It should be noted that none of them was informed as to the nature of the problem-solving questions. All ten participants' written works were described and analyzed based on the nature of the strategies employed in solving the problems. Four of the participants were interviewed to generate and probe their understanding of the division-of-fractions concept. These four were chosen based on the uniqueness of their responses in the problem-solving tasks and were selected to purposely provide varied levels of responses. Each interview probed participants' written responses to the problem-solving tasks. Toward the end of the interview, these preservice teachers were then asked to write a word problem for the expression $\frac{5}{7} \div \frac{1}{2}$.

Instrument

Five problem-solving questions and a problem-posing task were designed and used to elicit preservice teachers' understanding of division of fractions. Each of the five problem-solving questions shows different situations (categories) which define ways of interpreting division of fractions. These problem situations are referred in this study as (1) measurement, (2) partitive (equal share), (3) unit rate, (4) inverse of multiplication, and (5) inverse of a Cartesian product (as related to area). Sinicrope, Mick, and Kolb (2002) referred to them as categorization of problem types involving division of fractions which they adapted from the concept of whole-number division. Table 1 shows the five problems under each category that are used in this study.

Table 1
Problems involving Division of Fractions

Category	**Problem**
Measurement	**(Juice Intake Problem)** You have 3 ¼ cups of grapefruit juice. You take medicine each day and your doctor wants you to limit the amount of grapefruit juice you drink when you take your medicine. You are only allowed to drink ¾ cup of grapefruit juice each day with your breakfast. Can you drink the exact amount of grapefruit juice allowed without having to buy more? (Adapted from Sharp & Adams, 2002)
Partitive (Equal Share)	**(Ribbon Sharing Problem)** Four friends bought 2½ yards of ribbon. Is it enough to share equally the ribbon among them if each share is $\frac{2}{3}$ yard?
Unit Rate	**(Page Count Problem)** Alice reads the same number of pages of her book each day. After 8 days, she has $\frac{5}{6}$ of the book left to read. After another 4 days, she has 180 pages left to read. How many pages does Alice read in a day? How many pages are there in her storybook? (Adapted from Bai, 2003)
Inverse of Multiplication	**(Student Survey Problem)** In a student survey on class time preferences, 55 students said they prefer morning sessions. This is two and one-half times the number of students who prefer the afternoon sessions. How many students responded to the survey?
Inverse of Cartesian Product (in relation to area)	**(Plot Fencing Problem)** Is three meters of fencing enough to enclose a rectangular vegetable plot which has a length of $\frac{3}{4}$ of a meter and an area of $\frac{6}{20}$ square meter?

Some of these five problems were adapted from other research studies, as noted in Table 1, and were designed to assess conceptual and procedural understanding of the concept of division of fractions. All five problems require justification to elicit or provide additional information about students' problem-solving strategies and conceptual understanding. Requiring preservice teachers to justify their answer enables access to their thinking (Foong, 2002) and ensures that those who got it right really know the concepts behind the tasks (Gay & Thomas, 1993). Since these problem-solving tasks were designed for preservice teachers, special emphasis was given on contexts that deepen preservice teachers' knowledge of mathematics (Joyner & Bright, 1998). The preservice teachers were told to use any method they thought would best answer the problem. This request generated creative strategies but also demonstrated the preservice teachers' understanding of division of fractions.

Results from Preservice Teachers' Strategies

In the discussion which follows, findings regarding preservice teachers' strategies in solving division of fractions problems are discussed based on the five problem situations: measurement, partitive (equal share), unit rate, inverse of multiplication, and inverse of Cartesian product (as related to area). In particular, important ideas on fractions that are highlighted in these questions are identification of the number of groups, the number in a group, finding a unit rate, relating division with multiplication, and applying the concept of area.

Measurement. In this test item, students were assessed on whether or not they clearly understood this concept of division of fraction where the situation involved asking "how many groups are there?" In this case, however, the Juice Intake Problem (see Table 1) was specifically asking for a complete group.

All ten preservice teachers solved this problem correctly. Seven of them used pictorial solutions, two used tables, and only one used an arithmetic solution. From the responses of the four interviewed students, it was clear that they employed the measurement definition of division. However, their solution strategies varied. Patricia and Kelly employed pictorial solutions whereas Angela and Melody both set up a table using repeated subtraction and addition, respectively. All of them presented a sound argument for their solution.

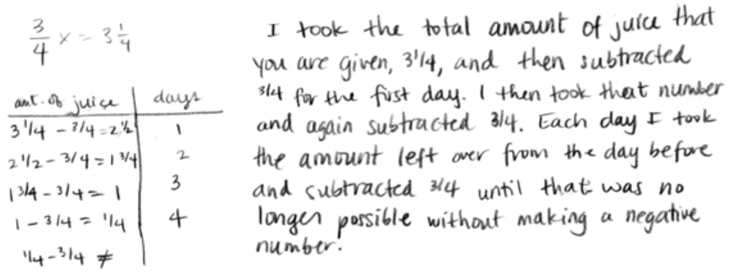


Figure 1: Angela's Repeated Subtraction

Angela considered the given amount of grapefruit juice first and successively subtracted the required amount to be taken everyday. Her strategy was common among students who view division as repeated subtraction. Also, her strategy focused on subtracting the amount required each day till the given amount of grapefruit juice was consumed. She reasoned, however, that there would not be enough amount of juice to be taken on the fourth day to meet the required juice intake. Thus, she concluded that “the exact amount of grapefruit juice cannot be drunk, without having to buy more.” It is interesting to notice that in her solution above, Angela set up the equation $\frac{3}{4}x = 3\frac{1}{4}$. However, her solution did not proceed by simply solving the equation using her algebra skills. Instead, she used repeated subtraction as a strategy to derive a solution.

Like Angela, Melody presented a tabular solution. However, instead of repeatedly subtracting she successively added the amount of juice to be taken in a day. Then, she stopped the process on the fifth day since she considered that the resulting amount could now be compared to the given amount of grapefruit juice. She wrote in her explanation, “I found that I could not drink the exact amount of grapefruit juice allowed without having to buy more because on day 4, I will have drunk a total of 3 cups which means that on day 5, I only have ¼ of a cup left, leaving me ½ of a cup short of the total amount of grapefruit juice I can drink for that day $(\frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2})$.” The figure below showed Melody’s repeated addition strategy.

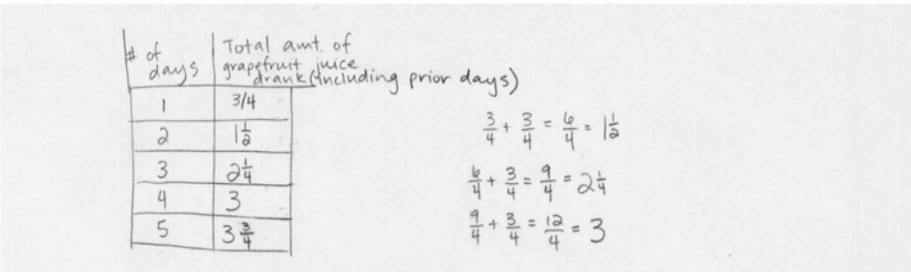


Figure 2: Melody’s Repeated Addition

Kelly and Patricia both used a pictorial solution involving similar analysis and strategy to solve the problem. For both, the solution was based on counting how many whole groups of ¾’s there were in 2¼; partitioning the given whole was employed to derived a solution. Though similar to Patricia’s solution strategy, Kelly was the only one among the ten preservice teachers in this study who had not taken a problem-solving course where pictorial representation and solution are emphasized. Her choice of strategy was, according to her, typical of her problem-solving strategies since she better understands when drawings are used as representations.

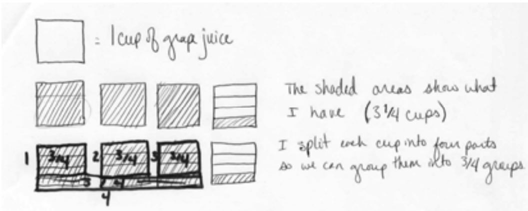


Figure 3: Kelly's Solution

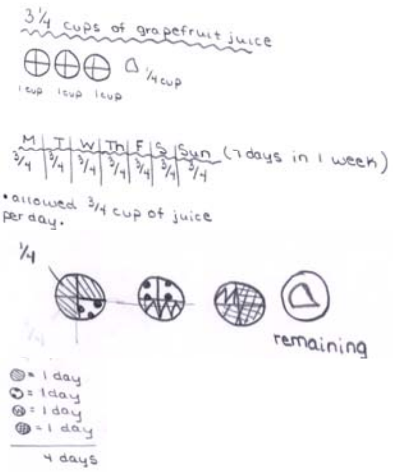
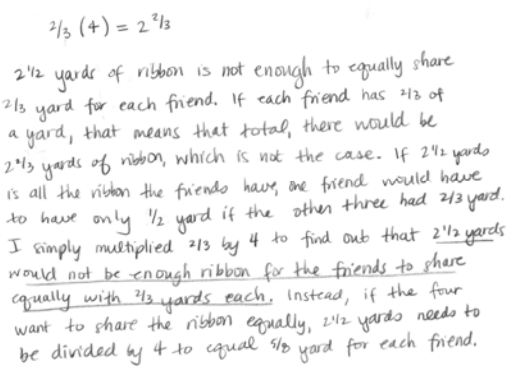


Figure 4: Patricia's Solution

In summary, from Angela, Melody, Kelly, and Patricia's solution strategies, it was clear that all employed the measurement definition of division; that is, looking for "how many groups there are". However, each solution was presented in a unique way. Also, all of them did not recognize right away that this problem was a division problem and thus the uniqueness of their strategies.

Partitive (Equal Share). When asked whether it is enough to share equally 2½ yards of ribbon among four friends who want to have a share of ⅔ yard each (Ribbon Sharing Problem), all ten preservice teachers correctly argued that the given amount of ribbon was not enough. Five preservice teachers used arithmetic and five used pictorial solutions. Two of the participants used the number line to solve this particular problem hence illustrating the students' attempts to use the measurement interpretation of division rather than the partitive one. On the other hand, Patricia and Kelly solved this problem using pictorial solutions by partitioning with their solutions showing a clear understanding of the partitive concept of division which interprets division as finding "how many in a group there are". Angela and Melody, on the other hand, used direct multiplication to get the amount if each is to have ⅔ of a yard. They both used the multiplicative strategy in solving this problem and viewed the problem as a non-division problem.


2/3 (4) = 2 2/3

2 1/2 yards of ribbon is not enough to equally share 2/3 yard for each friend. If each friend has 2/3 of a yard, that means that total, there would be 2 2/3 yards of ribbon, which is not the case. If 2 1/2 yards is all the ribbon the friends have, one friend would have to have only 1/2 yard if the other three had 2/3 yard. I simply multiplied 2/3 by 4 to find out that 2 1/2 yards would not be enough ribbon for the friends to share equally with 2/3 yards each. Instead, if the four want to share the ribbon equally, 2 1/2 yards needs to be divided by 4 to equal 5/8 yard for each friend.

Figure 5: Angela's Multiplicative Reasoning

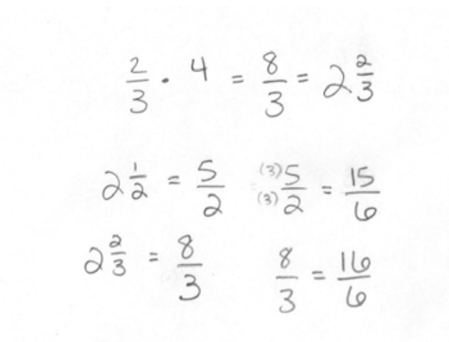


Figure 6: Melody's Multiplicative Solution

Consistent with their previous solution types, Kelly and Patricia again used pictorial solution for this type of problem. However, this time their strategy employed the measurement interpretation of division by counting how many full groups of $\frac{2}{3}$ there were in order to know whether there would be full groups of four representing the share for each of the four friends.

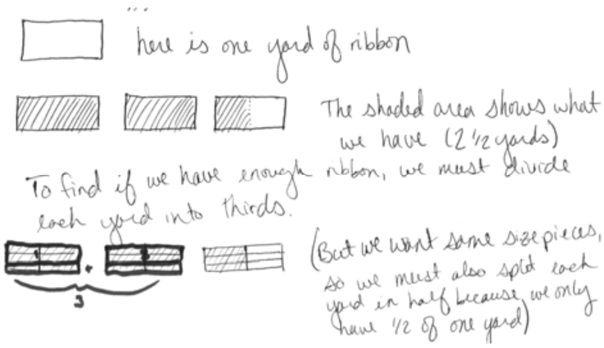


Figure 7.: Kelly's Solution

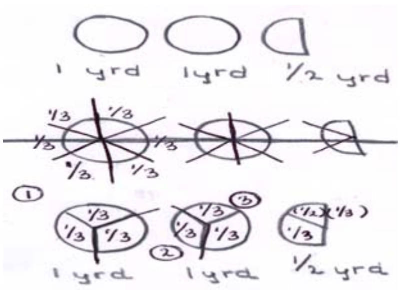


Figure 8. Patricia's Solution

Unit rate. One problematic aspect of the concept of division of fractions for preservice teachers is shown in their solution to the Page Count Problem. When asked to solve for the unit rate in this problem, all participants considered this task the most difficult among the five problems.

Angela, Kelly, and Melody all used a tabular solution for the problem and correctly solved the problem. However, for all of them it was not easy to derive the solution. They actually had to come back to the problem and did it as the last one to solve. They recognized that they spent more time thinking and rethinking about this task. Five of the participants solved the problem pictorially, one used arithmetic, one had no idea how to proceed, and three used a tabular solution. Ultimately, however, all but two of the participants got the correct solution.

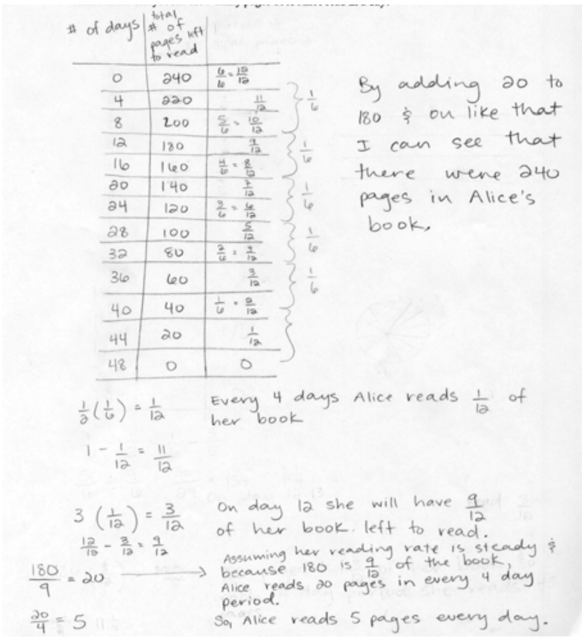


Figure 9. Melody's Solution

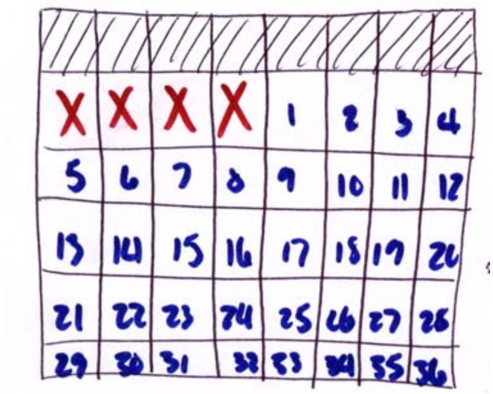


Figure 10. Kelly's Solution

Inverse of Multiplication. In the Student Survey Problem, two of the participants failed to see the meaning of the information given in the tasks. Patricia, in particular, had difficulty in seeing how to represent pictorially the values which, in this case, were fractions. Others failed to find the solution because of their inability to solve manually for fractions.

Patricia's inability to employ other methods of solving this problem aside from pictorial representation, disallowed her to see the simplicity of the problem. Angela, Kelly, and Melody used setting up the equation $2\frac{1}{2}x = 55$ and presented a sound argument for their solutions. In general, four preservice teachers recognized that this task was a division problem, so they directly performed division to show their solution. Three of them set up an equation which was basically their mathematical translation of the first statement in the problem. One still solved using pictures but failed to solve correctly. The other two failed to provide a correct and sound argument for this problem.

Inverse of Cartesian Product (in relations to area). When asked to decide if three meters of fencing was enough to enclose a rectangular vegetable plot which has a length of $\frac{3}{4}$ of a meter and an area of $\frac{6}{20}$ square meter (Plot Fencing Problem), all ten preservice teachers recognized the task in relation to the area formula. All of them immediately applied the concept of finding the perimeter given the area. Thus, all of them used the area formula to start solving the problem. Yet, four of them recognized the relationship to area but failed to present a complete and correct solution because of their inability to deal with fractions arithmetically. Patricia presented the formula yet again proceeded with the pictorial solution. Kelly, Melody, and Angela all used the area formula and presented a sound argument of their solution.

Table 2 summarizes the solution types employed, strategies used, and level of difficulty perceived by preservice teachers in solving the five problems. This table indicates that preservice teachers had difficulty solving problem involving finding the unit rate while they apparently could see the relationship between division as inverse of multiplication. Also, solving using a pictorial solution was common since most of them had been given instruction in solving pictorially. Finally, they had multiple ways of looking at division problems, not just limited to the algorithm of finding the quotient. Some explanation for this finding may be attributed to the fact that they were not informed about the type of problems given, and thus their strategies are more varied and less conventional.

Table 2
Summary of Results

Problems	**Fraction-Division Situations**	**Solution Types**	**Strategies Used**	**Perceived Level of Difficulty**[1]
Juice Intake	Measurement	Pictorial Tabular Arithmetic	Repeated addition Repeated subtraction	4th
Ribbon Sharing	Partitive (Equal Share)	Pictorial Arithmetic Number Line	Multiplicative Partitioning	3rd
Page Counting	Unit Rate	Pictorial Tabular Arithmetic	Finding parts of a whole Setting up an Equation	5^{th} (most difficult)
Student Survey	Inverse of Multiplication	Arithmetic Equation Pictorial	Direct division	1^{st} (easiest)
Plot Fencing	Inverse of Cartesian Product (in terms of area)	Equation	Area formula	2nd

[1](1 Easy to 5 Difficult)

Results from Problem Posing

When asked to write a word problem for the mathematical expression $\frac{5}{7} \div \frac{1}{2}$, Patricia, Kelly, Melody, and Angela all had difficulty writing a good word problem to represent this expression, though from the previous five tasks, most of them presented a good solution and a sound argument to the problem. The analysis of the problem-posing task suggested that their conceptual understanding of division of fractions is not solid. When asked to solve the expression, all four preservice teachers used the "invert and multiply rule" to reach a solution. As for the other solutions, a pictorial solution was a second choice. Most students failed to demonstrate an understanding of fraction division in coming up with distinct solution for this expression. For example, consider the word problems posed by Kelly and Melody for the expression $\frac{5}{7} \div \frac{1}{2}$. Each of these questions had problematic elements, which were suggestive of preservice teachers' weak conceptual understanding of any of the interpretations of division of fractions.

In response to the task, Kelly wrote:

> I have $\frac{5}{7}$ of a sandwich. My roommate wants half of what's left of my sandwich and I give it to her. How much do I have left?

Kelly's problem represented the expression, $\frac{5}{7}-\frac{1}{2}(\frac{5}{7})$, which is not what the task was asking. Her difficulty, like those of Patricia and Angela, was a good example of students' common misconception and confusion with the term "half of something" as something divided by $\frac{1}{2}$. That is, students often interpret or express a problem like this as division by two and not divided by one half.

In contrast, Melody wrote for her problem:

> Bob had two candy bars and wanted to share them with his friends. He gave Mary $\frac{5}{7}$ of one candy bar and he gave John $\frac{1}{2}$ of another. Bob meets Mario on the way to school and wants to give him the rest of his candy bars. How much would he have left to give him?

From Melody's representation it appears that she considered taking the fraction $\frac{5}{7}$ and $\frac{1}{2}$ from two different units which again not relate in any way to the expression $\frac{5}{7}\div\frac{1}{2}$. Melody's representation showed that she does not fully understand the concept of division of fractions.

Problems posed by Kelly, Patricia, and Angela were examples of most students' misconception regarding division of fractions. All three thought and claimed that the phrase half of five-sevenths meant $\frac{5}{7}\div\frac{1}{2}$. In these three cases, they were referring to $\frac{5}{7}\div 2$, not $\frac{5}{7}\div\frac{1}{2}$. They failed to describe a proper situation to express division by $\frac{1}{2}$. This case is consistent with Ball's (1990a) study where some preservice teachers perceived the expression $1\frac{3}{4}\div\frac{1}{2}$ as division by two instead of division by one half. Also, regarding division of fractions, Ball (1990a) found that most preservice teachers perceived the problems of division by fraction in terms of fractions and not as a division problem and they had difficulty relating the fractional expression to real life situations. Ball's finding is also supported by the result of this study. It is thus apparent and obvious for these cases why Ma (1999) suggested that a teacher should first have a profound mathematical understanding in order to have a pedagogically powerful representation of fractions. Kelly, Patricia, Angela, and Melody are particular cases of students who do no not have this profound understanding of division of fractions.

Conclusion

This study described preservice teachers' strategies and understanding of division of fractions. Findings supported the notion that preservice teachers should be taught with emphasis on understanding, both conceptual and procedural, to be able to competently face the challenge of teaching the subject matter. In particular, this study showed specific cases of preservice teachers' use-of-strategy in solving problems involving division of fractions and of their limited understanding of the concept of division of fractions. Results for the study showed that preservice teachers use several strategies in solving problems involving division of fractions. However, their ability to solve problems does not warrant conceptual understanding of the topic. Clearly, they can solve problems procedurally using different strategies but their conceptual understanding of division of fractions is not profound as shown in their inability to define correctly division from the problem-posing task.

These specific cases provide evidence, and are indicative, of curricular implications for the teaching and learning with understanding of preservice teachers. The findings and analysis are suggestive of the need to (1) provide a variety of problem-solving situations in teaching division of fractions; (2) promote the use of problem-posing questions in teaching; (3) encourage the use of multiple representations to solve a problem; and (4) design tasks that assess both procedural and conceptual understanding of division of fractions.

It is, however, recognized that the focus of this study is limited to its analysis of the preservice teachers' written work and short interviews and that more comprehensive analysis can be generated if preservice teachers were observed and monitored for a longer period of time. Investigating how mathematical understanding develops over a period of time is essential to capture the growth process of students' understanding. Thus, for further work, it is paramount to investigate the growth of preservice teachers' mathematical understanding across time and in the social context where learning occurs.

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Author:

Leah Nillas, Illinois State University, Normal, Illinois. nillas@lycos.com